Analytic Hierarchy Process

1. Introduction

The Analytic Hierarchy process was first created by Thomas Saaty in 1977. The model was designed to use pairwise comparisons, between both the options and the criteria used in the model, to create a model that could determine the optimal solution to a decision. The Ideal Analytic Hierarchy Process, a revised version of the Analytic Hierarchy Process, is what will be used in this model. While there are many different multi-criteria decision-making models that can be used, the Ideal Hierarchy Process is the least likely to give you an incorrect optimal solution.

2. User Input

In order to run the model, we will first need the user to input the number of parameters in the trade study (m) as well as the number of different options (n) that they are evaluating in the trade study. Next the user will have to input the names of the variables in *descending importance* (i.e. most important first). Once, the user has entered the names of the variables, the user will have to enter in the names of the options (in any order). The user will then have to enter the preferences for the parameters (1 if a higher score in the parameter is better, 0 if a lower score in the parameter is better).

The next item that the user will have to enter is a series of pairwise comparisons between two different parameters. The user will use the chart below to determine the scores that should be entered into each of the pairwise comparisons (Figure 1).

Scale for Pairwise Comparison	
Intensity of Importance	**Definition**
1/3	The second parameter is favored very strongly over the first parameter
1/2	The second parameter is favored strongly over the first parameter
2/3	The second parameter is favored moderately more than the first parameter
1	The first and second parameter are favored equally
3/2	The first parameter is favored moderately more than the second parameter
2	The first parameter is favored strongly over the second parameter
3	The first parameter is favored very strongly over the second parameter

Figure 1

Lastly, the user will have to enter in a matrix of raw data, **R[m][n]**, that is formatted in the same way as the diagram below (Figure 2).

For example, parameters m = 5, options n = 3

	Option 1	Option 2	Option 3
Parameter 1	r_{11}	r_{12}	r_{13}
Parameter 2	r_{21}	r_{22}	r_{23}
Parameter 3	r_{31}	r_{32}	r_{33}
Parameter 4	r_{41}	r_{42}	r_{43}
Parameter 5	r_{51}	r_{52}	r_{53}

Figure 2

3. Theory and Application

3.1 Parameter Weighting Matrix, P[m][m]

	Parameter 1	Parameter 2	Parameter 3	Parameter 4	Parameter 5
Parameter 1	1	p_{12}	p_{13}	p_{14}	p_{15}
Parameter 2	p_{21}	1	p_{23}	p_{24}	p_{25}
Parameter 3	p_{31}	p_{32}	1	p_{34}	p_{35}
Parameter 4	p_{41}	p_{42}	p_{43}	1	p_{45}
Parameter 5	p_{51}	p_{52}	p_{53}	p_{54}	1

Figure 3

The parameter weighting matrix ($\boldsymbol{P}[\boldsymbol{m}][\boldsymbol{m}]$), used to find the weights of the parameters, consist of a series of pairwise comparisons between parameters with respect to importance. There are some key properties that must exist within the matrix. The first property that must be noted is that the diagonal elements are all 1. This is due to the fact that those cells represent a pairwise comparison between the same parameters, so they should always equal 1. The next property that we must know is that $p_{ij} = \frac{1}{p_{ij}}\ for\ all\ i, j < m$. Thus traditionally, when using the Ideal Analytic Hierarchy Process, the user would have to input all of the pairwise comparisons for the blue and green cells (Figure 3) by using Saaty's scale. However, when dealing with a relatively large amount of parameters, achieving *consistency* is almost impossible. Inconsistency occurs when you have a contradiction in the relative importance of a parameter or option. An example of inconsistency would be saying that $p_{14} > p_{15}$ and $p_{15} > p_{12}$, but $p_{12} > p_{14}$. To deal with this problem, we will employ a new method of entering the pairwise comparisons.

To help understand how to get a consistent matrix, we will create bijective mapping from Saaty's Scale (Figure 4) to a new scale (Figure 5).

The Fundamental Scale for Pairwise Comparisons		
Intensity of Importance	Definition	Explanation
1	Equal importance	Two elements contribute equally to the objective
3	Moderate importance	Experience and judgment moderately favor one element over another
5	Strong importance	Experience and judgment strongly favor one element over another
7	Very strong importance	One element is favored very strongly over another; its dominance is demonstrated in practice
9	Extreme importance	The evidence favoring one element over another is of the highest possible order of affirmation
Intensities of 2, 4, 6, and 8 can be used to express intermediate values. Intensities of 1.1, 1.2, 1.3, etc. can be used for elements that are very close in importance.		

Figure 4

New Scale for Pairwise Comparison	
Intensity of Importance	Definition
1	Equal Importance
3	One element is 3 times as important as another element
5	One element is 5 times as important as another element
7	One element is 7 times as important as another element
9	One element is 9 times as important as another element

Figure 5

The mapping ($f: X \rightarrow Y$) can be visualized by the diagram below (Figure 6) (Note that only a few of the elements will be mapped and that the mapping could be different due to the subjectivity of Saaty's scale).

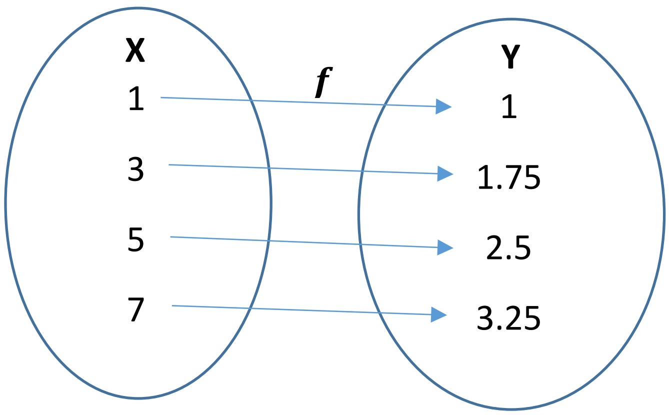

Figure 6

Since f is bijective, we can also note that the inverse function of f exists and is well defined.

We will now consider a 4x4 matrix that is made according to the new scale.

$$\begin{bmatrix} 1 & a & d & f \\ \frac{1}{a} & 1 & b & e \\ \frac{1}{d} & \frac{1}{b} & 1 & c \\ \frac{1}{f} & \frac{1}{e} & \frac{1}{c} & 1 \end{bmatrix}$$

We can get the following set of equalities by applying the definition of the scales (Elements have parenthesis around them).

$(1) = a(2) \qquad (1) = d(3) \qquad (1) = f(4) \qquad (2) = b(3) \qquad (2) = e(4) \qquad (3) = c(4)$

We can use the above equalities to create a new equality.

$$(1) = a(2) \Rightarrow (1) = ab(3) \Rightarrow (1) = abc(4)$$

Similarly:

$$(1) = ae(4)$$

$$(1) = dc(4)$$

$$(1) = f(4)$$

We can see that in order for the matrix to be consistent, the following must be true: $bc = e$, $dc = f$, and $d = ab$.

Thus, in order for our matrix to be consistent we must have:

$$\begin{bmatrix} 1 & a & ab & abc \\ \frac{1}{a} & 1 & b & bc \\ \frac{1}{ab} & \frac{1}{b} & 1 & c \\ \frac{1}{abc} & \frac{1}{bc} & \frac{1}{c} & 1 \end{bmatrix}$$

We can now transform our matrix using the inverse of the function f:

$$\begin{bmatrix} 1 & f^{-1}(a) & f^{-1}(ab) & f^{-1}(abc) \\ \left(\frac{1}{f^{-1}(a)}\right) & 1 & f^{-1}(b) & f^{-1}(bc) \\ \left(\frac{1}{f^{-1}(ab)}\right) & \left(\frac{1}{f^{-1}(b)}\right) & 1 & f^{-1}(c) \\ \left(\frac{1}{f^{-1}(abc)}\right) & \left(\frac{1}{f^{-1}(bc)}\right) & \left(\frac{1}{f^{-1}(c)}\right) & 1 \end{bmatrix}$$

Using the antidistributive property, we get:

$$\begin{bmatrix} 1 & f^{-1}(a) & f^{-1}(a)f^{-1}(b) & f^{-1}(a)f^{-1}(b)f^{-1}(c) \\ \left(\frac{1}{f^{-1}(a)}\right) & 1 & f^{-1}(b) & f^{-1}(b)f^{-1}(c) \\ \left(\frac{1}{f^{-1}(a)f^{-1}(b)}\right) & \left(\frac{1}{f^{-1}(b)}\right) & 1 & f^{-1}(c) \\ \left(\frac{1}{f^{-1}(a)f^{-1}(b)f^{-1}(c)}\right) & \left(\frac{1}{f^{-1}(b)f^{-1}(c)}\right) & \left(\frac{1}{f^{-1}(c)}\right) & 1 \end{bmatrix}$$

Thus, it is evident that in order to achieve consistency in Saaty's scale, the matrix must look like the one above.

Since it is almost impossible for a user to achieve consistency or near consistency, especially in large matrices, it may be beneficial to prompt the user for the highlighted values in the matrix above. In order to keep the elements of the matrix somewhat bounded, we must also restrict the user to using values from 1/3 to 3 (figure 1). A user may enter a higher (or lower) value vary rarely if desired, however, it may cause an unintentional trivial situation (i.e. one parameter has almost all the weight). It is also suggested that the user ranks the parameters in order of importance before they start. This will get rid of a situation where the user has to compare the most important parameter to the least important parameter and is forced to only put a 3, even though the number should be higher.

Now that we have a consistent weighting matrix, we can create the weights. We will first take the geometric mean of the rows of the weighting matrix (Figure 3). The geometric mean for $p_{i1}, p_{i2}, \dots, p_{im}$ would be $\sqrt[m]{p_{i1} \times p_{i2} \times \cdots \times p_{im}}$. We will then standardize each geometric mean by dividing by the sum of the geometric means. For example if you had geometric means $g_1, g_2, \dots, g_m$, then the standardized mean of g_1 would be $\frac{g_1}{g_1+g_2+\cdots+g_m}$, which would give us w_1. Thus, we will end up with our parameters weighting vector $\boldsymbol{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$.

3.2 Consistency

As stated earlier, inconsistency occurs when the user inputs values into the weighting matrix (or option matrices). A Matrix that is perfectly consistent will always have a Consistency Index (CI) = 0. If the matrix has too much inconsistency the model will not run correctly, and the user is more likely to get an "optimal" solution that is not the true optimal solution. To determine how much inconsistency is acceptable, we will compare the CI to the Random Index (RI) (Figure 7). It was determined by Saaty, that if $\frac{CI}{RI} > 0.10$ then the matrix will require further analysis and should probably be reconstructed to create a more consistent matrix.

m	2	3	4	5	6	7	8	9	10
RI	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.51

Figure 7

To calculate the CI for a matrix, we must estimate λ^{max}, the maximum eigenvalue. To estimate λ^{max} (for the weighting matrix) we must multiply P (weighting matrix) by $\boldsymbol{w}$ (weighting vector) . Let $\boldsymbol{P}^{(m \times m)} \times \boldsymbol{w}^{(m \times 1)} = \boldsymbol{v}^{(m \times 1)}$. We can then approximate λ^{max} with $\hat{\lambda}_i^{max} = \frac{\boldsymbol{v}_i}{\boldsymbol{w}_i}$. Our final approximate of λ^{max}, $\hat{\lambda}^{max}$, will be the arithmetic mean (average) of $\hat{\lambda}_i^{max}$ $for\ i = 1, \dots, m$. Thus we can approximate CI by $\widehat{\text{CI}} = \frac{(\hat{\lambda}^{max} - m)}{m-1}$. However, since we use a revised method of inputting the pairwise comparisons, we will not have to worry about inconsistency in the weighting matrix (yet the ratio will still be calculated to catch any errors).

3.3 Option Weight Matrices, O[n][n][m]

With respect to parameter k

	Option 1	Option 2	Option 3
Option 1	**1**	$\boldsymbol{o}_{12}$	$\boldsymbol{o}_{13}$
Option 2	$\boldsymbol{o}_{21}$	**1**	$\boldsymbol{o}_{23}$
Option 3	$\boldsymbol{o}_{31}$	$\boldsymbol{o}_{32}$	**1**

Raw Data With respect to parameter k

	Option 1	Option 2	Option 3
Option 1	$\boldsymbol{r}_{k1}$	$\boldsymbol{r}_{k2}$	$\boldsymbol{r}_{k3}$
Option 2	$\boldsymbol{r}_{k1}$	$\boldsymbol{r}_{k2}$	$\boldsymbol{r}_{k3}$
Option 3	$\boldsymbol{r}_{k1}$	$\boldsymbol{r}_{k2}$	$\boldsymbol{r}_{k3}$

Figure 8

Similarly to the weighting matrix, the option matrices ($\boldsymbol{O}^{(k)}$) are a set of matrices that consist of pairwise comparisons. There will be m matrices where each matrix will be $n \times n$ in size. Figure 8 (above) illustrates one of the matrices. In this matrix, b_{12} represents the comparison of option 1 to option 2 with respect to the k^{th} parameter. We will use Saaty's scale in order to determine the value that corresponds to that pairwise comparison. However, consider a trade study with 10 parameters and 10 different options. The user would have to fill out 10 different matrices, and the matrices would probably not be consistent. Since we have the raw data for each of the options, we will explore a method which will use the raw data to create all the matrices.

In order to create a good model, $\boldsymbol{O}^{(\boldsymbol{k})}$ should have b_{ij} that vary greatly. The greater variation will help distinguish the utility scores between the different options. At the same time, we want to try and bound the scores between $\frac{1}{9}$ and 9. One proposed way to do this was to take the range (user specified) and to divide the range into 17 sections (one for each integer or fraction) and then assign scores based off of where the options fall. The major downfall is that the scores will be most likely evenly distributed, which would not be ideal. When ranking things against one another, most options will be similar in performance, so they should have a higher probability of being closer to 1 and a relatively low probability of being $\frac{1}{9}$ or 9. Thus if we normalize the option scores, we will find that most of the scores will be close to 1, however, that does not stop an option from being a 8 or 9.The premise of this idea comes from Chebyshev's inequality.

Chebyshev's inequality states that for any distribution where the mean and the variance are defined, we are guaranteed that the following percent of observations fall within n standard deviations of the mean:

n	percentage
2	75%
3	88.9%
4	93.75%
5	96%
6	97.22%
10	99%

Let a higher score in a parameter be preferable to a lower score (Preference = 1).

Consider $o_{ij} = \begin{cases} 1 + \left(\frac{r_{ii}-r_{ij}}{\sigma}\right) \; if \; r_{ii} \geq r_{ij} \\ \frac{1}{1+\left(\frac{r_{ij}-r_{ii}}{\sigma}\right)} \; if \;\; r_{ii} < r_{ij} \end{cases}$.

This ensure that the majority of the o_{ij}'s are close to one, however it gives more weight when there are distinct differences between the two raw data points. The advantage to this method is that it does not matter what size of a scale you are working with, because the standard deviation will change between small scales and large scales. While it is unlikely, if $o_{ij} > 9$ the model will still run, however it will give more weight than what is intended to that pairwise comparison of options. This method also helped create consistency in the option matrices, so we do not have to worry about consistency in the option matrices.

If a lower score is preferable when compared to a higher score (Preference = 0),

Consider $o_{ij} = \begin{cases} 1 + \left(\frac{r_{ij}-r_{ii}}{\sigma}\right) \; if \; r_{ii} \leq r_{ij} \\ \frac{1}{1+\left(\frac{r_{ii}-r_{ij}}{\sigma}\right)} \; if \;\; r_{ii} > r_{ij} \end{cases}$.

If a median score in preferable for parameter i, say s, it is recommended that you replace the raw data with $|r_{ij} - s|$, and then use the preference = 0.

Now, we have consistent option matrices that are ready to be used in the model. Similarly to the weight matrix, we will take the geometric mean of each row and standardize them. Thus for $\boldsymbol{O}^{(k)}$ we will get a vector say $\boldsymbol{s}^{(l)} = \begin{bmatrix} s_{1l} \\ \vdots \\ s_{nl} \end{bmatrix}$. Thus we will create a new matrix, $\boldsymbol{S}^{(n \times m)}$, from our vectors $\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)}, \ldots\ \boldsymbol{s}^{(m)}$, such that $\boldsymbol{S} = [\boldsymbol{s}^{(1)} \vdots\ \boldsymbol{s}^{(2)} \vdots\ \ldots \vdots\ \boldsymbol{s}^{(m)}]$. Since we are using the Ideal Analytic Process instead of the original model, we will take the max s_{jl} in each column, and divided each other entry in that column by the maximum entry. Thus one column may look like $\boldsymbol{s}^{(l)} = \begin{bmatrix} \frac{s_{1l}}{s_{jl}} \\ \vdots \\ \frac{s_{nl}}{s_{jl}} \end{bmatrix}$ (If s_{jl} is the maximum value in that column). We will call this standardized new matrix $\boldsymbol{S}^{*}$. We will also check each of the option matrices consistency in a similar fashion as the weighting matrix. The diagram below (Figure 9) shows a visual representation of what is happening throughout this section.

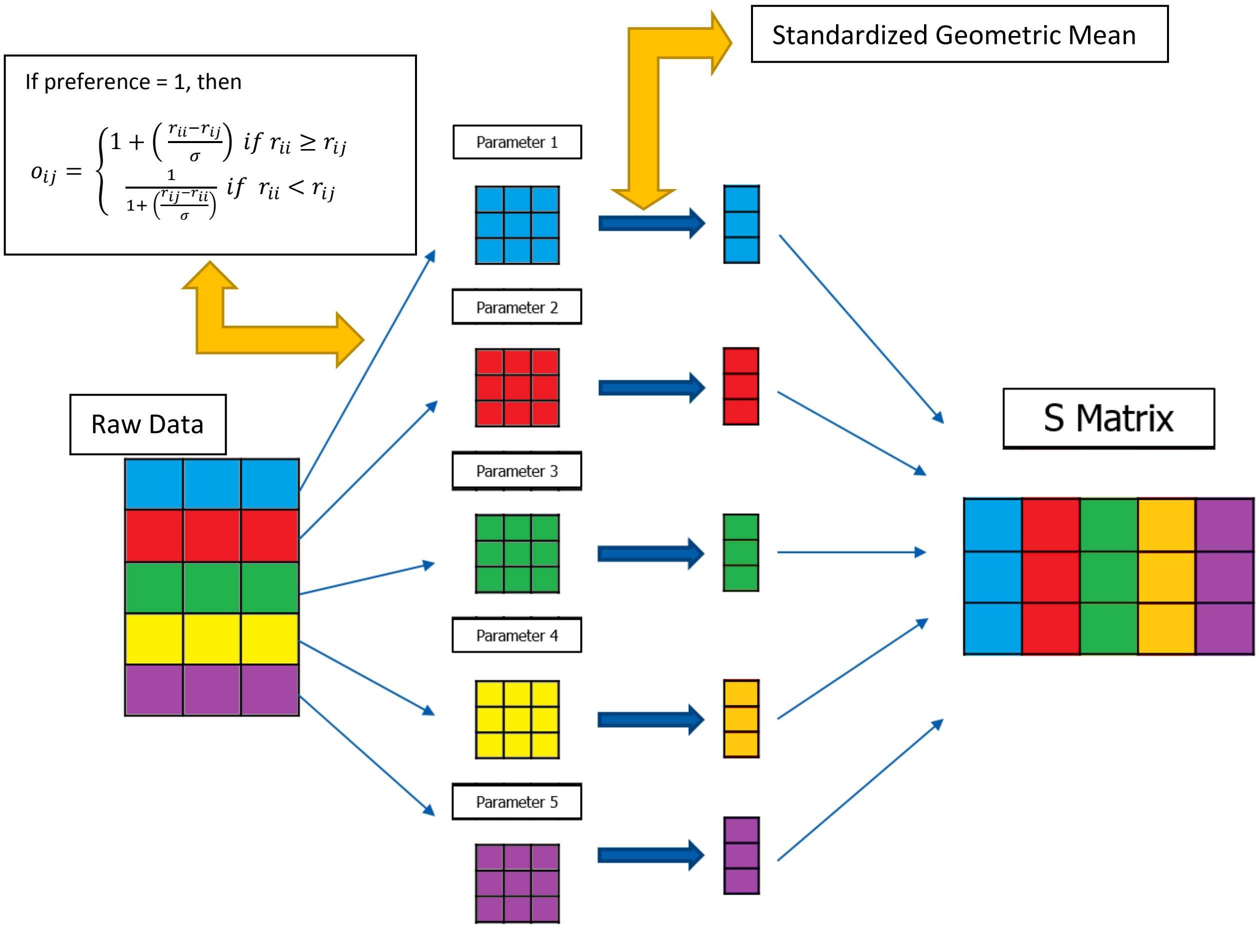

Figure 9

3.4 Finding the Utility Score, S[n][1]

Thus far, we have computed the weighting vector, $\boldsymbol{w}$, and have constructed $\boldsymbol{S}^*$ from the option matrices. Lastly to find the Utility Scores for each of the options, we will multiply $\boldsymbol{S}^* \times \boldsymbol{w}$. Therefore we will have a new vector of size n. We will call this vector $\boldsymbol{y}$. Next we will normalize $\boldsymbol{y}$ by diving by the sum of the elements, and get our new vector, say S. The first entry will correspond to the first option and will continue in the order that the options have been inputted into the model. We must note that these scores are relative, and that they add up to 1. Thus if you add a new option, the average will go down, making it seem like the options got worse, when in fact that is not the case. Thus once the scores are computed, the optimal option will be the one with the highest score.

3.5 Sensitivity

Sensitivity is an important aspect of this analysis, and will be calculated after the utility scores have been calculated. Since the scores are computed off of raw data, and they are in a way out of our control, sensitivity analysis on the option matrices are not crucial. Instead we will perform sensitivity analysis on the weights for the parameters. The sensitivity analysis will tell you what the percent change would have to be in the weighting of parameters in order for another option to become the optimal solution. The following is an example of computing the sensitivity of option 1 (the current optimal solution) vs option 2, with respect to parameter 1.

We can set up an equation to solve for the needed weight:

$\big(\boldsymbol{y}\,(1) - \boldsymbol{y}(2)\big) - \boldsymbol{w}(1)\big(\boldsymbol{S}(1,1) - \boldsymbol{S}(1,2)\big) = h(\big(\boldsymbol{S}(1,1) - \boldsymbol{S}(1,2)\big)$ where h is the new weight for $\boldsymbol{w}(1)$

Once we solve for h, we can find the percent change of the weight:

$$\frac{h - \boldsymbol{w}(1)}{\boldsymbol{w}(1)} \times 100$$

We can conclude that if all are large percent change of the weight needed for one option to become the optimal option, that the optimal solution is relatively stable. In other words, if you change the relative importance of the parameters by a little bit, you will still have the same optimal solution. On the other hand, if the percent change is relatively small, then further analysis must be done between the two options.

The format of the sensitivity analysis will be as followed:

Assume option 1 (O_1) is the optimal solution.

	Parameter 1	Parameter 2	Parameter 3	Parameter 4	Parameter 5
$O_1 - O_2$	%	%	%	%	%
$O_1 - O_3$	%	%	%	%	%
$O_1 - O_4$	%	%	%	%	%
$O_1 - O_5$	%	%	%	%	%

4. Numerical Example

For this Numerical Example, we will be using the data from PBW_Access_Trade_Fs_TRB_8-9-16. Unlike the analysis before, we will be using Cost and Risk as parameters in the study. We will say that Risk and Cost will be equally important, Cost will be favored strongly over Equipment Accessibility, Equipment Accessibility is favored moderately more than Flexibility/ Adaptability, Flexibility/ Adaptability is equally as important as Compliance with R&R Time Requirements, Compliance with R&R Time Requirements is favored moderately more than Modular Manufacturing /Integration, and Modular Manufacturing /Integration is favored moderately more than Ease of In-Silo Maintenance. Thus we fill in the weighting matrix $\boldsymbol{P}$ as followed (Note: the tan boxes will be calculated):

1	1	2	3	3	4.5	6.75
1	1	2	3	3	4.5	6.75
.5	.5	1	1.5	1.5	2.25	3.375
.333	.333	.667	1	1	1.5	2.25
.333	.333	.667	1	1	1.5	2.25
.222	.222	.444	.667	.667	1	1.5
.148	.148	.296	.444	.444	.667	1

Thus we get a weighting vector that looks like the following:

.2827
.2827
.1414
.0942
.0942
.0628
.0419

Next we will approximate the eigenvalue:

$$(1 \times .2827) + (1 \times .2827) + (2 \times .1414) + (3 \times .0942) + (3 \times .0942) + (4.5 \times .0628) + (6.75 \times .0419) = 1.9789, \qquad \frac{1.9789}{.2827} = 7 \rightarrow \hat{\lambda}_1^{max} = 7$$

We will do this for the rest of the rows of P and get $\hat{\lambda}^{max} = 7$.

Thus $\widehat{\mathrm{CI}} = \frac{(7-7)}{7-1} = 0$. Since M = 7, RI = 1.32. Therefore $\frac{CI}{RI} = 0$, so the matrix is perfectly consistent.

Next we will move on to the Option Matrices.

Our Raw Data looks like:

1	1	1	2	2	2
10800	78500	19700	0	600	19400
9	6	6	6	6	9
8	2	6	8	8	8
8	1	8	6	1	8
8	8	6	8	8	8
6	6	6	4	4	8

We will note that since we want a low cost and low risk, our preference vector will be:

$$[\,0\ 0\ 1\ 1\ 1\ 1\ 1]$$

Using the first row (highlighted green), we can get the option matrix with respect to the parameter Risk:

1	1	1	2.8257	2.8257	2.8257
1	1	1	2.8257	2.8257	2.8257
1	1	1	2.8257	2.8257	2.8257
.3539	.3539	.3539	1	1	1
.3539	.3539	.3539	1	1	1
.3539	.3539	.3539	1	1	1

Taking the standardized geometric means of the rows, we get:

1.6810
1.6810
1.6810
.5949

.5949
.5949

We can do that for all 7 matrices and come up with $\boldsymbol{S}$

1.680994	1.202992	2.050623	1.360745	1.562122	1.229209	1.151269
1.680994	0.367196	0.698324	0.370342	0.49457	1.229209	1.151269
1.680994	0.968561	0.698324	0.787573	1.562122	0.356345	1.151269
0.594886	1.558879	0.698324	1.360745	1.072505	1.229209	0.527981
0.594886	1.536671	0.698324	1.360745	0.49457	1.229209	0.527981
0.594886	0.975707	2.050623	1.360745	1.562122	1.229209	2.350894

We can determine $\boldsymbol{S}^{*}$by dividing all the elements in the column by the largest element of that column:

1	0.771703	1	1	1	1	0.489716
1	0.235551	0.340542	0.272161	0.316601	1	0.489716
1	0.621319	0.340542	0.57878	1	0.289898	0.489716
0.353889	1	0.340542	1	0.68657	1	0.224587
0.353889	0.985754	0.340542	1	0.316601	1	0.224587
0.353889	0.625903	1	1	1	1	1

Similarly to the parameter matrix, we will check all of the option matrices consistency. In this case, all the matrices are consistent.

Next we will find our Utility Scores by multiplying $\boldsymbol{S}^{*}$by $\boldsymbol{w}$ to get:

0.220726
0.129498
0.16759
0.159877
0.150485
0.171823

Thus we can see that 2nd Diving Board; Two-Side Access is our optimal option. We can notice that when we add in parameters such as cost and risk that the optimal option becomes a lot clearer and there exists more separation. In the excel file, the sixth option was actually better than the first option, whereas when cost and risk are accounted for, we can see that the first option is clearly superior.

We will now look at the stability matrix:

Inf	-249.238	-405.271	-550.792	-586.61	Inf	Inf
Inf	-517.553	-236.049	-554.333	Inf	-493.231	Inf
-137.948	390.4123	-270.312	Inf	-853.106	Inf	-2269.19
-159.24	480.6659	-312.034	Inf	-451.655	Inf	-2619.43
-110.865	-491.298	Inf	Inf	Inf	Inf	947.5332

We can see that smallest percentage is -110.865. However, that would mean the parameter weight would have to be negative, which is not allowed in this model. Note that these are the percentages needed if you were to only change the weight on one parameter. Thus if you change the weight on numerous parameters, these confidence Thus we can say that this outcome is stable and you could change the weights slightly and still come out with the same optimal answer.