The Analytic Hierarchy Process

The *Analytic Hierarchy Process* (AHP), introduced by Thomas Saaty (1980), is an effective tool for dealing with complex decision making, and may aid the decision maker to set priorities and make the best decision. By reducing complex decisions to a series of pairwise comparisons, and then synthesizing the results, the AHP helps to capture both subjective and objective aspects of a decision. In addition, the AHP incorporates a useful technique for checking the consistency of the decision maker's evaluations, thus reducing the bias in the decision making process.

1 How the AHP works

The AHP considers a set of evaluation criteria, and a set of alternative options among which the best decision is to be made. It is important to note that, since some of the criteria could be contrasting, it is not true in general that the best option is the one which optimizes each single criterion, rather the one which achieves the most suitable trade-off among the different criteria.

The AHP generates a weight for each evaluation criterion according to the decision maker's pairwise comparisons of the criteria. The higher the weight, the more important the corresponding criterion. Next, for a fixed criterion, the AHP assigns a score to each option according to the decision maker's pairwise comparisons of the options based on that criterion. The higher the score, the better the performance of the option with respect to the considered criterion. Finally, the AHP combines the criteria weights and the options scores, thus determining a global score for each option, and a consequent ranking. The global score for a given option is a weighted sum of the scores it obtained with respect to all the criteria.

2 Features of the AHP

The AHP is a very flexible and powerful tool because the scores, and therefore the final ranking, are obtained on the basis of the pairwise relative evaluations of both the criteria and the options provided by the user. The computations made by the AHP are always guided by the decision maker's experience, and the AHP can thus be considered as a tool that is able to translate the evaluations (both qualitative and quantitative) made by the decision maker into a multicriteria ranking. In addition, the AHP is simple because there is no need of building a complex expert system with the decision maker's knowledge embedded in it.

On the other hand, the AHP may require a large number of evaluations by the user, especially for problems with many criteria and options. Although every single evaluation is very simple, since it only requires the decision maker to express how two options or criteria compare to each other, the load of the evaluation task may become unreasonable. In fact the number of pairwise comparisons grows quadratically with the number of criteria and options. For instance, when comparing 10 alternatives on 4 criteria, $4 \cdot 3/2=6$ comparisons are requested to build the weight vector, and $4 \cdot (10 \cdot 9/2)=180$ pairwise comparisons are needed to build the score matrix.

However, in order to reduce the decision maker's workload the AHP can be completely or partially automated by specifying suitable thresholds for automatically deciding some pairwise comparisons.

3 Implementation of the AHP

The AHP can be implemented in three simple consecutive steps:

1) Computing the vector of criteria weights.

2) Computing the matrix of option scores.
3) Ranking the options.

Each step will be described in detail in the following. It is assumed that m evaluation criteria are considered, and n options are to be evaluated. A useful technique for checking the reliability of the results will be also introduced.

3.1 Computing the vector of criteria weights

In order to compute the weights for the different criteria, the AHP starts creating a *pairwise comparison matrix* **A**. The matrix **A** is a $m \times m$ real matrix, where m is the number of evaluation criteria considered. Each entry a_{jk} of the matrix **A** represents[1] the importance of the jth criterion relative to the kth criterion. If $a_{jk} > 1$, then the jth criterion is more important than the kth criterion, while if $a_{jk} < 1$, then the jth criterion is less important than the kth criterion. If two criteria have the same importance, then the entry a_{jk} is 1. The entries a_{jk} and a_{kj} satisfy the following constraint:

$$a_{jk} \cdot a_{kj} = 1. \tag{1}$$

Obviously, $a_{jj} = 1$ for all j. The relative importance between two criteria is measured according to a numerical scale from 1 to 9, as shown in Table 1, where it is assumed that the jth criterion is equally or more important than the kth criterion. The phrases in the “Interpretation” column of Table 1 are only suggestive, and may be used to translate the decision maker’s qualitative evaluations of the relative importance between two criteria into numbers. It is also possible to assign intermediate values which do not correspond to a precise interpretation. The values in the matrix **A** are by construction pairwise consistent, see (1). On the other hand, the ratings may in general show slight inconsistencies. However these do not cause serious difficulties for the AHP.

Value of a_{jk}	***Interpretation***
1	j and k are equally important
3	j is slightly more important than k
5	j is more important than k
7	j is strongly more important than k
9	j is absolutely more important than k

Table 1. Table of relative scores.

Once the matrix **A** is built, it is possible to derive from **A** the *normalized pairwise comparison matrix* $\mathbf{A}_{norm}$ by making equal to 1 the sum of the entries on each column, i.e. each entry $\bar{a}_{jk}$ of the matrix $\mathbf{A}_{norm}$ is computed as

$$\bar{a}_{jk} = \frac{a_{jk}}{\sum_{l=1}^{m} a_{lk}}. \tag{2}$$

Finally, the *criteria weight vector* ***w*** (that is an m-dimensional column vector) is built by averaging the entries on each row of $\mathbf{A}_{norm}$, i.e.

$$w_j = \frac{\sum_{l=1}^{m} \bar{a}_{jl}}{m}. \tag{3}$$

[1] For a matrix **A**, a_{ij} denotes the entry in the ith row and the jth column of **A**. For a vector **v**, v_i denotes the ith element of **v**.

3.2 *Computing the matrix of option scores*

The matrix of option scores is a $n \times m$ real matrix **S**. Each entry s_{ij} of **S** represents the score of the *i*th option with respect to the *j*th criterion. In order to derive such scores, a *pairwise comparison matrix* $\mathbf{B}^{(j)}$ is first built for each of the *m* criteria, $j=1,...,m$. The matrix $\mathbf{B}^{(j)}$ is a $n \times n$ real matrix, where *n* is the number of options evaluated. Each entry $b_{ih}^{(j)}$ of the matrix $\mathbf{B}^{(j)}$ represents the evaluation of the *i*th option compared to the *h*th option with respect to the *j*th criterion. If $b_{ih}^{(j)} > 1$, then the *i*th option is better than the *h*th option, while if $b_{ih}^{(j)} < 1$, then the *i*th option is worse than the *h*th option. If two options are evaluated as equivalent with respect to the *j*th criterion, then the entry $b_{ih}^{(j)}$ is 1. The entries $b_{ih}^{(j)}$ and $b_{hi}^{(j)}$ satisfy the following constraint:

$$b_{ih}^{(j)} \cdot b_{hi}^{(j)} = 1 \tag{4}$$

and $b_{ii}^{(j)} = 1$ for all *i*. An evaluation scale similar to the one introduced in Table 1 may be used to translate the decision maker's pairwise evaluations into numbers.

Second, the AHP applies to each matrix $\mathbf{B}^{(j)}$ the same two-step procedure described for the pairwise comparison matrix **A**, i.e. it divides each entry by the sum of the entries in the same column, and then it averages the entries on each row, thus obtaining the score vectors $\boldsymbol{s}^{(j)}$, $j=1,...,m$. The vector $\boldsymbol{s}^{(j)}$ contains the scores of the evaluated options with respect to the *j*th criterion.

Finally, the score matrix **S** is obtained as

$$\mathbf{S} = [\ \boldsymbol{s}^{(1)} \ldots \boldsymbol{s}^{(m)}\ ] \tag{5}$$

i.e. the *j*th column of **S** corresponds to $\boldsymbol{s}^{(j)}$.

Remark. In the considered DSS structure, the pairwise option evaluations are performed by comparing the values of the performance indicators corresponding to the decision criteria. Hence, this step of the AHP can be considered as a transformation of the indicator matrix **I** into the score matrix **S**.

3.3 *Ranking the options*

Once the weight vector ***w*** and the score matrix **S** have been computed, the AHP obtains a vector **v** of global scores by multiplying **S** and **w**, i.e.

$$\boldsymbol{v} = \mathbf{S} \cdot \boldsymbol{w} \tag{6}$$

The *i*th entry v_i of **v** represents the global score assigned by the AHP to the *i*th option. As the final step, the option ranking is accomplished by ordering the global scores in decreasing order.

4 Checking the consistency

When many pairwise comparisons are performed, some inconsistencies may typically arise. One example is the following. Assume that 3 criteria are considered, and the decision maker evaluates that the first criterion is *slightly* more important than the second criterion, while the second criterion is *slightly* more important than the third criterion. An evident inconsistency arises if the decision maker evaluates by mistake that the third criterion is equally or more important than the first criterion. On the other hand, a slight inconsistency arises if the decision maker evaluates that the first criterion is also *slightly* more important than the third criterion. A consistent evaluation would be, for instance, that the first criterion is more important than the third criterion.

The AHP incorporates an effective technique for checking the consistency of the evaluations made by the decision maker when building each of the pairwise comparison matrices involved in the process, namely the matrix **A** and the matrices $\mathbf{B}^{(j)}$. The technique relies on the computation of a suitable *consistency index*, and will be described only for the matrix **A**. It is straightforward to adapt it to the case of the matrices $\mathbf{B}^{(j)}$ by replacing **A** with $\mathbf{B}^{(j)}$, $\boldsymbol{w}$ with $\boldsymbol{s}^{(j)}$, and m with n. The *Consistency Index* (*CI*) is obtained by first computing the scalar x as the average of the elements of the vector whose jth element is the ratio of the jth element of the vector $\mathbf{A}\cdot\boldsymbol{w}$ to the corresponding element of the vector $\boldsymbol{w}$. Then,

$$CI = \frac{x-m}{m-1}. \tag{7}$$

A perfectly consistent decision maker should always obtain *CI*=0, but small values of inconsistency may be tolerated. In particular, if

$$\frac{CI}{RI} < 0.1 \tag{8}$$

the inconsistencies are tolerable, and a reliable result may be expected from the AHP. In (8) *RI* is the *Random Index*, i.e. the consistency index when the entries of **A** are completely random. The values of *RI* for small problems ($m \leq 10$) are shown in Table 2.

m	2	3	4	5	6	7	8	9	10
RI	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.51

Table 2. Values of the *Random Index* (*RI*) for small problems.

The matrices **A** corresponding to the cases considered in the above example are shown below, together with their consistency evaluation based on the computation of the consistency index. Note that the conclusions are as expected.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1/3 \\ 1/3 & 1 & 3 \\ 3 & 1/3 & 1 \end{bmatrix} \quad \Rightarrow \text{CI/RI} = 1.150 \quad \Rightarrow \text{inconsistent}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ 1/3 & 1 & 3 \\ 1/3 & 1/3 & 1 \end{bmatrix} \quad \Rightarrow \text{CI/RI} = 0.118 \quad \Rightarrow \text{slightly inconsistent}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 1/3 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix} \quad \Rightarrow \text{CI/RI} = 0.033 \quad \Rightarrow \text{consistent}$$

5 *Automating the pairwise comparisons*

Although every single AHP evaluation is very simple (the decision maker is only required to express how two criteria or alternatives compare to each other), the load of the evaluation task may become unreasonable and tedious for the decision maker when many criteria and alternatives are considered. However, in order to alleviate the decision maker's workload, some pairwise comparisons can be completely or partially automated. A simple method is suggested in the following.

Let the *j*th criterion be expressed by an attribute which assumes values in the interval [$I_{j,\min}$, $I_{j,\max}$], and let $I_j^{(i)}$ and $I_j^{(h)}$ be the instances of the attribute under the *i*th and *h*th control options, respectively. Assume that the larger the value of the attribute, the better the system performance according to the *j*th criterion. If $I_j^{(i)} \geq I_j^{(h)}$, the element $b_{ih}^{(j)}$ of $\mathbf{B}^{(j)}$ can be computed as

$$b_{ih}^{(j)} = 8\frac{I_j^{(i)} - I_j^{(h)}}{I_{j,\max} - I_{j,\min}} + 1. \quad (10)$$

A similar expression holds if the smaller the value of the attribute, the better the system performance according to the *j*th criterion. If $I_j^{(i)} \leq I_j^{(h)}$, the element $b_{ih}^{(j)}$ of $\mathbf{B}^{(j)}$ can be computed as

$$b_{ih}^{(j)} = 8\frac{I_j^{(h)} - I_j^{(i)}}{I_{j,\max} - I_{j,\min}} + 1. \quad (11)$$

Note that (10) and (11) are linear functions of the difference $I_{ij} - I_{hj}$. Of course, More sophisticated functions can be designed by exploiting specific knowledge and/or experience.

6 An illustrative example

An example will be here described in order to clarify the mechanism of the AHP. *m*=3 evaluation criteria are considered, and *n*=3 alternatives are evaluated. Each criterion is expressed by an attribute. The larger the value of the attribute, the better the performance of the option with respect to the corresponding criterion. The decision maker first builds the following pairwise comparison matrix for the three criteria:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 1/3 & 1 & 3 \\ 1/5 & 1/3 & 1 \end{bmatrix}$$

to which corresponds the weight vector $\boldsymbol{w}$ = [0.633 0.261 0.106]^T. Then, based on the values assumed by the attributes for the three options (see Figure 1), the decision maker builds the following pairwise comparison matrices:

$$\mathbf{B}^{(1)} = \begin{bmatrix} 1 & 3 & 7 \\ 1/3 & 1 & 5 \\ 1/7 & 1/5 & 1 \end{bmatrix}, \quad \mathbf{B}^{(2)} = \begin{bmatrix} 1 & 1/5 & 1 \\ 5 & 1 & 5 \\ 1 & 1/5 & 1 \end{bmatrix}, \quad \mathbf{B}^{(3)} = \begin{bmatrix} 1 & 5 & 9 \\ 1/5 & 1 & 3 \\ 1/9 & 1/3 & 1 \end{bmatrix}$$

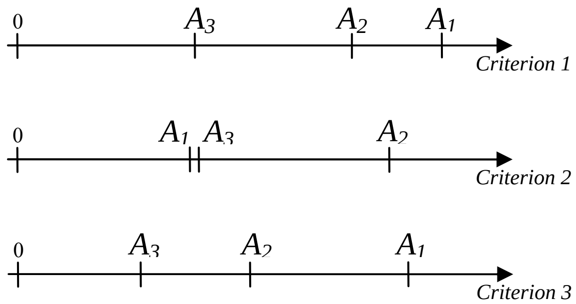

Figure 1. Values of the attributes for the alternatives A_1, A_2 and A_3 (the scale on each axis is not relevant).

to which correspond the score vectors $\boldsymbol{s}^{(1)}$ = [0.643 0.283 0.074]^T, $\boldsymbol{s}^{(2)}$ = [0.143 0.714 0.143]^T, and $\boldsymbol{s}^{(3)}$ = [0.748 0.180 0.072]^T.

Hence, the score matrix **S** is

$$\mathbf{S} = [\boldsymbol{s}^{(1)} \quad \boldsymbol{s}^{(2)} \quad \boldsymbol{s}^{(3)}] = \begin{bmatrix} 0.643 & 0.143 & 0.748 \\ 0.283 & 0.714 & 0.180 \\ 0.074 & 0.143 & 0.072 \end{bmatrix}$$

and the global score vector is $\boldsymbol{v} = \mathbf{S} \cdot \boldsymbol{w}$ = [0.523 0.385 0.092]^T. Note that the first option turns out to be the most preferable, though it is the worst of the three with respect to the second criterion.

References

Saaty, T.L., 1980. “The Analytic Hierarchy Process.” McGraw-Hill, New York.